グラフ理論に基づいたLattice fermionと Laplacian operatorとしてのWilson term

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Background (Species and Wilson term)

格子上の場の理論には，fermion doublingという難問が存在する。

Fermion doubling

QCDに必要な性質を理論に課すことで，
複数のlattice fermion (**species**)が出現するという問題。

これらのspeciesは縮退しているため、区別することができない。

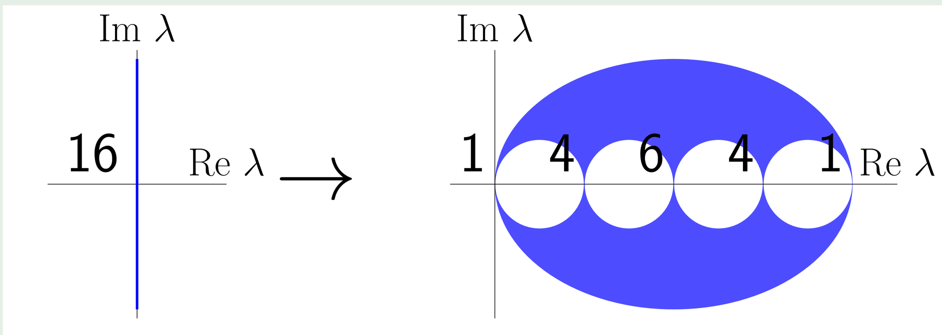
Wilson fermion : species-splitting mass fermion

Wilson term : species-splitting term $S_W = \sum_{n,\mu} \frac{a^5}{2} \bar{\psi}_n (2\psi_n - \psi_{n+\mu} - \psi_{n-\mu})$

$$D_W(p) = \frac{1}{a} \sum_\mu \left[i\gamma_\mu \sin a p_\mu + (1 - \cos a p_\mu) \right]$$

Physical $(0,0,0,0)$:
$D_W(p) = i\gamma_\mu p_\mu + O(a)$

Non physical $(\pi/a, 0, 0, 0)$:
$D_W(p) = i\gamma_\mu p_\mu + \frac{2}{a} + O(a)$



Previous research

Maximal # of species depend on the topological invariant of lattice.

Table: Topological invariant and maximum # of the species

lattice	Topological invariant	maximal # of species d
4-d torus	$1+4+6+4+1$	16
Torus T^D	$(1+1)^D$	2^D
Hyperball B^D	$1+0+0+\cdots$	1
Sphere S^D	$1+0+0+\cdots+1$	2
$T^D \times B^d$	2^D+0	2^D

数学的な説明を与えるために、graph theoryに着目した。

Lattice fermions as graph theory

of Fermion species $\Longrightarrow$ Nullity of spectral matrix [J.Y, T.MIsumi (2022)]
Wilson term $\Longrightarrow$ Laplacian matrix and topological invariant

New conjecture on species doubling of lattice fermions!

Spectral graph theory

Definition (graph)

A graph G is a pair $G=(V,E)$. V is a set of vertices and E is a set of edge.

Definition (directed graph)

A directed graph is a pair (V,E) of sets of vertices and edges together with two maps $\mathrm{init}: E \to V$ and $\mathrm{ter}: E \to V$. The two maps are assigned to every edge e_{ij} with an initial vertex $\mathrm{init}(e_{ij}) = v_i \in V$ and a terminal vertex $\mathrm{ter}(e_{ij}) = v_j \in V$. If $\mathrm{init}(e_{ij}) = \mathrm{ter}(e_{ij})$, the edge e_{ij} is called a loop.

Definition (weighted graph)

A weighted graph has a value (weight) for each edge in a graph.

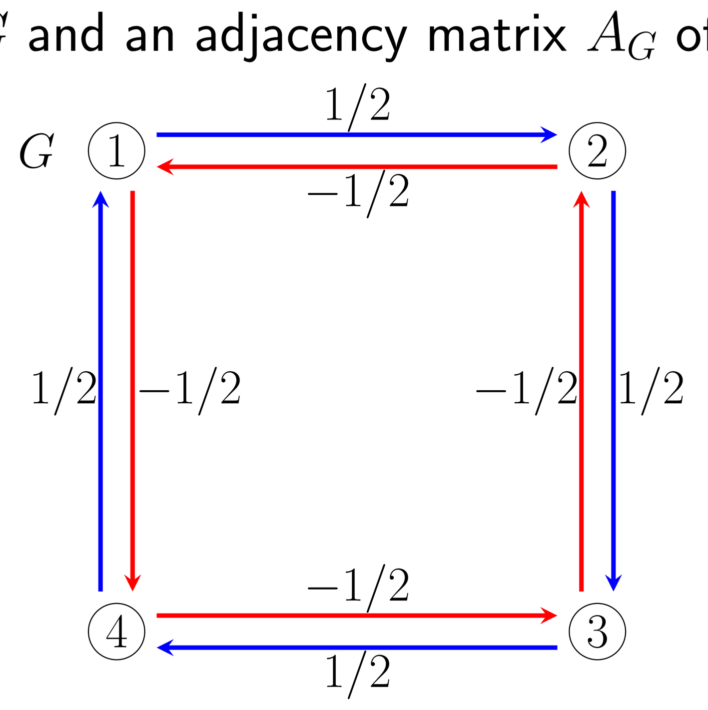
Definition (adjacency matrix)

An adjacency matrix A of graph is the $|V| \times |V|$ matrix given by

$$A_{ij} \equiv \begin{cases} w_{ij} & \text{if there is a edge from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

where w_{ij} is the weight of an edge from i to j.

e.g. A graph G and an adjacency matrix A_G of G.



$$A_G = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

Lattice fermions as spectral graphs

J.Y, T.Misumi, JHEP **02**, 104 (2022).

The naive lattice action as spectral graph theory is

$$S = \sum_{m,n} \sum_\mu \bar{\psi}_m \gamma_\mu D_{mn} \psi_n = \sum_\mu \bar{\boldsymbol{\psi}} \gamma_\mu A_\mu \boldsymbol{\psi} \equiv \bar{\boldsymbol{\psi}} \mathcal{D} \boldsymbol{\psi}$$

where $\boldsymbol{\psi} = \sum_n \psi_n |n\rangle$ with $|n\rangle \equiv \bigotimes_\mu |n_\mu\rangle$.

$$\mathcal{D}(k) = \sum_{k,\mu} i\gamma_\mu \sin\left[\frac{2\pi}{N_\mu}(k_\mu - 1)\right] |k\rangle\langle k| = 0 \quad \Rightarrow \quad \sin\left[\frac{2\pi}{N_\mu}(k_\mu - 1)\right] = 0$$

16 species!

Lattice field theory $\Longrightarrow$ Spectral graph theory
Lattice fermion $\Longrightarrow$ Directed and Weighted spectral graph
# of Fermion species $\Longrightarrow$ Nullity of spectral matrix

Wilson term as Laplacian operator

Definition (Laplacian operator)

A Laplacian operator L of a graph $G(V,E)$ is given by

$$L_{ij} \equiv \begin{cases} d_i & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } (i,j) \text{ are linked} \\ 0 & \text{otherwise} \end{cases}$$

where d_i is the number of edges sharing the site i.

e.g. The Lagrangian on T^4 constructed from the Laplacian operator $\mathcal{L}$ is

$$\bar{\boldsymbol{\psi}} \mathcal{L} \boldsymbol{\psi} = 2 \sum_{n,\mu} \bar{\psi}_n (2\psi_n - \psi_{n+\mu} - \psi_{n-\mu}) \quad \Rightarrow \quad S_W = \frac{a^5}{4} \bar{\boldsymbol{\psi}} \mathcal{L} \boldsymbol{\psi}$$

The Laplacian operator results in the Wilson term!

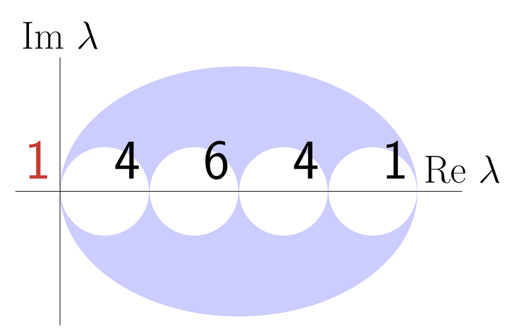
Theory (Hodge theory)

The nullity of a Laplacian operator defined on a complex chain coincides with the 0-th Betti number that is the topological invariant.

e.g. The Betti number and nullity of Dirac matrix with the Laplacian on T^4

the r-th Betti number β_r of T^4

$\beta_0(T^4) = 1, \quad \beta_1(T^4) = 4, \quad \beta_2(T^4) = 6,$
$\beta_3(T^4) = 4, \quad \beta_4(T^4) = 1,$



The nullity of Dirac matrix with the Laplacian is consistent with Hodge theory, so # of zero-eigenvalue $= \beta_0(T^4) = 1$.
The r-th Betti number and each species-splitting number just match!

Wilson term $\Longrightarrow$ Laplacian operator and topological invariant (Betti number)

Summary

Lattice fermions as graph theory

of Fermion species $\Longrightarrow$ Nullity of spectral matrix
Wilson term $\Longrightarrow$ Laplacian matrix and the Betti number

Lattice field theory

Graph theory ⟷ Topology

We believe that these are closely related.

Discussion

- A new conjecture on the number of fermion species on the discretized manifold.
  J.Y, T.Misumi [arXiv:2301.09805]
- The proof of the new conjecture.
- The Laplacian operator as Persistent homology and Persistent spectral graph.